Special Relativity III

Momentum & Energy

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Overview

Review
Momentum
Kinetic Energy
Momentum & Energy
Mass & Energy
Recap: Lectures 1&2
Postulates of Special Relativity

Special Relativity (1905)

The Relativity Postulate: The basic laws of physics are the same in all inertial reference frames.

Speed of Light Postulate: The speed of light in vacuum has the same value $c$ in all inertial reference frames.

Albert Einstein (1879-1955) at age 26.

From the Emilio Segré Visual Archives
The Lorentz Factor

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ v = c \sqrt{1 - \gamma^{-2}} \]

Limiting Cases

\[ \lim_{v \to 0} \gamma = 1 \]

\[ \lim_{v \to c} \gamma = \infty \]
Lorentz Transformations

For two inertial reference frames where the second frame moves at \( \mathbf{v} = v_0 \hat{i} \) with respect to the first.

\[
\begin{align*}
t' &= \gamma(t - vx/c^2) & t &= \gamma(t' + vx'/c^2) \\
x' &= \gamma(x - vt) & x &= \gamma(x' + vt') \\
y' &= y & z &= z'
\end{align*}
\]
Lorentz Transformations

For two inertial reference frames where the second frame moves at $\mathbf{v} = v_0 \hat{i}$ with respect to the first.

\[
\begin{align*}
  x^2 + y^2 &= c^2 t^2 \\
  x'^2 + y'^2 &= c^2 t'^2 \\
  t' &= \gamma(t - vx/c^2) \\
  x' &= \gamma(x - vt) \\
  y' &= y \\
  t &= \gamma(t' + vx'/c^2) \\
  x &= \gamma(x' + vt') \\
  z &= z'
\end{align*}
\]
Moving clocks run slow as measured by stationary observers.

\[ \Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} = \gamma \Delta t_0 \]

\( \Delta t \) elapsed time on a stationary observers clock.

\( \Delta t_0 \) elapsed time on a moving clock as measured by the stationary observers.

\[ \Delta t \geq \Delta t_0 \]
Length (Space) Contraction

Moving objects are **contracted** in the direction of motion as measured by a **stationary** observer.

\[ L = L_0 \sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0}{\gamma} \]

- \( L \) = length of a moving object as measured by a stationary observer.
- \( L_0 \) = length of the object when it is at rest.
Spacetime Diagrams

Spacetime Interval

\[(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2\]

Spacelike: \((\Delta s)^2 < 0\)

Timelike: \(\Delta s^2 > 0\)

Absolute Future

Absolute Past
The Lightcone in Two Frames

\[ t' = \gamma(t - vx/c^2) \]
\[ x' = \gamma(x - vt) \]

\[ t = \gamma(t' + vx'/c^2) \]
\[ x = \gamma(x' + vt') \]
Spacetime Diagrams

\[ t' = \gamma(t - vx/c^2) \]

\[ x' = \gamma(x + vt) \]

\[ t = \gamma(t' + vx'/c^2) \]

\[ x = \gamma(x' + vt') \]

\( \beta = 1/5 \)
Spacetime Diagrams

\[ t' = \gamma(t - vx/c^2) \]

\[ x' = \gamma(x - vt) \]

\[ t = \gamma(t' + vx'/c^2) \]

\[ x = \gamma(x' + vt') \]
\[ \beta = \frac{v}{c} \]

\[ f = \sqrt{\frac{1 \pm \beta}{1 \mp \beta}} f_0 \]  

Longitudinal

\[ f = \sqrt{1 - \beta^2} f_0 \]  

Transverse

General \( (0 \leq \beta < 1) \)

\[ f = \frac{1 \pm \beta \cos \theta'}{\sqrt{1 - \beta^2}} f_0 \]  

\( \theta' \) angle between light and \( x' \) axis.
Momentum

Newton's Second Law
Inelastic Collision
Relativistic Momentum
Momentum for $0 \leq u < c$

Kinetic Energy
Momentum & Energy
Mass & Energy
Newton’s Second Law

Newton II was invariant under Galilean transformations

\[ F = \frac{dp}{dt} = m\frac{d^2x}{dt^2} \]

Since \( a = a' \) under Galilean transformations. Computing \( a'_x \) from the previous Lorentz and velocity transformation relations:

\[ a'_x = \frac{du'_x}{dt'} = \frac{d}{dt'} \left( \frac{u_x - v}{1 - vu_x/c^2} \right) = \frac{a_x}{\gamma^3(1 - vu_x/c^2)} \neq a_x \]

What is the correct relativistic generalization of the Newtonian definition of momentum \( p = mv \)?
Inelastic Collision

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\begin{align*}
\text{before} & \\
& m \quad u_1 = v \quad m \quad u_2 = -v \\
& \text{after} \\
& m \quad m \quad V = 0
\end{align*}

\begin{align*}
\text{before} & \\
& m \quad m \quad u'_2 \\
& \text{after} \\
& m \quad m \quad V' = -v
\end{align*}
Inelastic Collision

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\[
\begin{align*}
\text{before} & \quad u_1 = v \\ 
\text{after} & \quad V = 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{before} & \quad \begin{aligned}
& m \quad u_1 = v \\
& m \quad u_2 = -v
\end{aligned} \\
\text{after} & \quad \begin{aligned}
& m \quad m \\
& V = 0
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\frac{u_2 - v}{1 - u_2 v/c^2} &= -\frac{2v}{1 + v^2/c^2} \\
\end{align*}
\]

\[
\begin{align*}
\text{before} & \quad \begin{aligned}
& m \quad m \\
& u_2' = -\frac{u_2 - v}{1 - u_2 v/c^2}
\end{aligned} \\
\text{after} & \quad \begin{aligned}
& m \quad m \\
& V' = -v
\end{aligned}
\end{align*}
\]
Relativistic Momentum

Using proper time

\[ p = m \frac{dx}{d\tau} = m \frac{dx}{dt} \frac{dt}{d\tau} = m u \gamma \]

or

\[ p = \gamma m u = \frac{m u}{\sqrt{1 - u^2/c^2}} \]

Relativistic form of Newton II:

\[ F = \frac{dp}{dt} = \frac{d}{dt} (\gamma m u) \]
Momentum for $0 \leq u < c$

**Newton II**

\[ F = \frac{dp}{dt} \]

Momentum

\[ p = \gamma mu \quad \text{Newtonian} \]

\[ p = \gamma mu \quad \text{Relativistic} \]

Lorentz Factor

\[ \gamma = \frac{1}{\sqrt{1 - u^2/c^2}} \]
Kinetic Energy

Momentum & Energy

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Relativistic Kinetic Energy

Newton II, with relativistic $\mathbf{p}$

$$
F = \frac{d\mathbf{p}}{dt} = \frac{d}{dt} (\gamma m \mathbf{u}) = \frac{d}{dt} \left( \frac{m \mathbf{u}}{\sqrt{1 - u^2/c^2}} \right)
$$

Work done on an object initially at rest:

$$
W = K = \int \mathbf{F} \cdot d\mathbf{s} = \int \frac{d}{dt} (\gamma m \mathbf{u}) \cdot \mathbf{u} dt
= m \int_{0}^{t_f} u dt \frac{d}{dt} (\gamma u) = m \int_{0}^{u_f} u d(\gamma u)
= m \int_{0}^{u_f} \left(1 - \frac{u^2}{c^2}\right)^{-3/2} u du
= m \left\{ \frac{c^2}{\sqrt{1-u^2/c^2}} \right\}_{0}^{u_f} = (\gamma - 1) mc^2
$$
\[ d(\gamma u) = d \left( \frac{u}{\sqrt{1-u^2/c^2}} \right) \]
\[ = \left( \frac{1}{\sqrt{1-u^2/c^2}} + \frac{u^2/c^2}{(1-u^2/c^2)^{3/2}} \right) du \]
\[ = \gamma \left( 1 + \frac{u^2/c^2}{(1-u^2/c^2)^{3/2}} \right) du \]
\[ = \gamma \left( \frac{1}{1-u^2/c^2} \right) du \]
\[ = (1 - u^2/c^2)^{-3/2} du \]
Correspondence to Newtonian Physics

In Einstein’s relativity, the kinetic energy of a particle is:

\[ \text{KE} = (\gamma - 1)mc^2 \]

For \( v \ll c \):

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \approx 1 + \frac{1}{2} \frac{v^2}{c^2} \]

Active Learning: Show that the kinetic energy reduces to that of Newtonian physics when \( v \ll c \).
**Kinetic Energy for** $0 \leq u < c$

**Energy**

$$E = K + mc^2 = \gamma mc^2$$

**Kinetic Energy**

- **Newtonian**
  $$K = \frac{1}{2} mu^2$$
- **Relativistic**
  $$K = (\gamma - 1)mc^2$$

**Lorentz Factor**

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$
Momentum & Energy

Review

Momentum

Kinetic Energy

Momentum & Energy

$E_0 = mc^2$ in his own words

Energy into Mass

Total Energy & Rest Energy

Energy (massive particles)

Energy and Momentum

$p\overline{p}$ collisions at Fermilab

Pion Decay

Mass & Energy
"It followed from the special theory of relativity that mass and energy are both but different manifestations of the same thing – a somewhat unfamiliar conception for the average mind. Furthermore, the equation \( E = mc^2 \) is equal to \( mc^2 \)-squared, in which energy is put equal to mass, multiplied by the square of the velocity of light, showed that very small amounts of mass may be converted into a very large amount of energy and vice versa. The mass and energy were in fact equivalent, according to the formula mentioned before. This was demonstrated by Cockcroft and Walton in 1932, experimentally."
$E_0 = mc^2$ Energy into Mass

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Pion Decay

Mass & Energy

Example

Pair production

$\gamma + \gamma \rightarrow e^+ + e^-$

“In Paris in 1933, Irène and Frédéric Joliot-Curie took a photograph showing the conversion of energy into mass. The first photograph showing the creation of a pair of particles, revealed by the fog spots they make in passing through the wet air of a ”cloud chamber.” The two particles, curving apart under the influence of a magnet, were created in the annihilation of a particle of light (coming invisibly from below).”

Source: http://www.aip.org/history/einstein/ae22.htm
Total Energy & Rest Energy

Kinetic Energy:

\[ E_K = \gamma mc^2 - mc^2 \]

Einstein’s Hypothesis:

\[ E = \gamma mc^2 \text{ (total energy)}, \quad E_0 = mc^2 \text{ (rest energy)} \]

Total energy is conserved:

\[ E = E_K + E_0 \]

Rest energy is invariant:

\[ E^2 - (pc)^2 = E'^2 - (p'c)^2 = (mc^2)^2 \]
Energy (massive particles)

Energy Relation

\[ E = E_0 + KE \]

\[ E = \gamma mc^2 \text{ total} \]

\[ KE = (\gamma - 1)mc^2 \text{ kinetic} \]

\[ E_0 = mc^2 \text{ rest} \]
Energy and Momentum

\[ E = \gamma mc^2, \quad p = \gamma mu, \quad \gamma^2 = \frac{1}{1 - u^2/c^2} \]

\[ E^2 - (pc)^2 = \gamma^2 m^2 c^4 \left( 1 - \frac{u^2}{c^2} \right) = (mc^2)^2 \]

**General Relationship**

\[ E^2 = (pc)^2 + (mc^2)^2 \]
Energy and Momentum

General Relationship

\[ E^2 = (pc)^2 + (mc^2)^2 \]

Massive Particles, \( v < c \)

\[
\begin{align*}
p &= \gamma mv \\
E &= \gamma mc^2
\end{align*}
\]

Massless particles, \( v = c \)

\[ E = pc \]
The Tevatron Accelerator at Fermilab can accelerate protons and anti-protons to energies of $E = 0.980 \text{ TeV} \approx 1 \text{ TeV} = 10^{12} \text{ eV}$.

Active Learning: What is the speed of a 1 TeV proton?  

NB $E_0 = m_p c^2 = 938 \text{ MeV}$, $1 \text{ MeV} = 10^6 \text{ eV}$. 

$E = m \times c^2$ in his own words

$E_0 = mc^2$

Energy into Mass

Total Energy & Rest Energy

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Momentum and Energy

$p \bar{p} \text{ collisions at Fermilab}$

Pion Decay

Mass & Energy
**$p \bar{p}$ collisions at Fermilab**

The Tevatron Accelerator at Fermilab can accelerate protons and anti-protons to energies of $E = 0.980$ TeV $\approx 1$ TeV $= 10^{12}$ eV.

**Active Learning:** What is the speed of a 1 TeV proton?

NB $E_0 = mpc^2 = 938$ MeV, $1$ MeV $= 10^6$ eV.

\[
E = \gamma mc^2 = \gamma E_0
\]

\[
\gamma = \frac{E}{E_0} = \frac{10^{12} \text{ eV}}{0.938 \times 10^9 \text{ eV}} = 1.07 \times 10^3
\]

\[
v/c = \sqrt{1 - \gamma^{-2}} \approx 1 - \frac{1}{2} \gamma^{-2} = 0.999999956
\]
Pion Decay: $\pi^- \longrightarrow \mu^- + \bar{\nu}_\mu$ (99.99%)

In the rest frame of the pion:

\[ E = m_\pi c^2 = E_\mu + E_{\bar{\nu}} \]

\[ p_\pi = 0 = p_\mu + p_{\bar{\nu}} \]

Particle Data Group: http://pdg.lbl.gov

$m_\pi = 139.57 \text{ MeV}/c^2$, $m_\mu = 105.66 \text{ MeV}/c^2$, $\tau_\pi = 2.6 \times 10^{-8} \text{ s}$
Pion Decay: $\pi^- \longrightarrow \mu^- + \bar{\nu}_\mu$ (II)

In General

$$E = \sqrt{(pc)^2 + (mc^2)^2}, \quad (pc) = \sqrt{E^2 - (mc^2)^2}$$

Conservation of $E$ and $p$

$$E = m_\pi c^2 = E_\mu + E_{\bar{\nu}}, \quad p_\pi = 0 = p_\mu + p_{\bar{\nu}}$$

In the pion rest frame:

$$\frac{(cp_\mu)^2}{E_\mu^2 - (m_\mu c^2)^2} = \frac{(cp_{\bar{\nu}})^2}{E_{\bar{\nu}}^2 - (m_{\bar{\nu}} c^2)^2}$$

$$E_\mu^2 - (m_\mu c^2)^2 = [m_\pi c^2 - E_\mu]^2 - (m_{\bar{\nu}} c^2)^2$$
Pion Decay: $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$  (III)

In the pion rest frame:

$$E_\mu = \frac{(m_\pi^2 + m_\mu^2 - m_{\bar{\nu}}^2)c^2}{2m_\pi}$$

$$E_{\bar{\nu}} = \frac{(m_\pi^2 + m_{\bar{\nu}}^2 - m_\mu^2)c^2}{2m_\pi}$$

Momenta in pion rest frame:

$$p_{\mu}^2 = p_{\bar{\nu}}^2 = \frac{m_\pi^4 + m_\mu^4 + m_{\bar{\nu}}^4 - 2m_\pi^2 m_\mu^2 - 2m_\pi^2 m_{\bar{\nu}}^2 - 2m_\mu^2 m_{\bar{\nu}}^2}{4m_\pi^2}c^2$$
Subscript symmetry: $\mu \leftrightarrow \bar{\nu}$

If $m_\mu = m_{\bar{\nu}}$:

$$E_\mu = E_\nu = \frac{1}{2}m_\pi c^2$$

If $m_\mu + m_{\bar{\nu}} = m_\pi \Rightarrow p_\mu = p_\nu = 0$
Binding Energy & Disintegration Energy

\[ E_0 = mc^2 \]


Bound system of mass \( M \) and constituents \( m_i \):

\[
E_B = \sum_i m_i c^2 - M c^2
\]

Disintegration energy \( (M > \sum_i m_i) \):

\[
Q = M c^2 - \sum_i m_i c^2 = \Delta mc^2
\]
Hydrogen and constituents

Hydrogen is composed of an electron and a proton. The mass of hydrogen is slightly less than the sum of the mass of its constituents.

Hydrogen Mass

\[ m_H = 938.78 \text{ MeV}/c^2 \]

Mass defect

\[ m_p + m_e - m_H = 13.6 \text{ eV}/c^2 \]

\( \Delta m \) is one part in 10^8, but

\[ E = mc^2 \]
Helium and Hydrogen (Deuterium)

Fusion

\[ D + D \rightarrow \text{He} + \text{energy} \]

Mass defect

\[ 2m_D - m_{\text{He}} = 0.0256 \, \text{u} = 23.8 \, \text{MeV}/c^2 \]

\( \Delta m \) is 6.4 parts in 10\(^3\).

\( 1 \, \text{u} = 1 \) unified mass unit = 931.494 MeV/c\(^2\)
Deuteron Binding Energy

Find the Binding Energy of the Deuteron given:

- **Proton** \( E_0 = 1.007276 \, c^2 \, u = 938.27 \, \text{MeV} \)
- **Neutron** \( E_0 = 1.008665 \, c^2 \, u = 939.57 \, \text{MeV} \)
- **Deuteron** \( E_0 = 2.01355 \, c^2 \, u = 1875.61 \, \text{MeV} \)
Deuteron Binding Energy

Find the Binding Energy of the Deuteron given:

\[
\begin{align*}
\text{Proton} & \quad E_0 = 1.007276 \, c^2 u = 938.27 \, \text{MeV} \\
\text{Neutron} & \quad E_0 = 1.008665 \, c^2 u = 939.57 \, \text{MeV} \\
\text{Deuteron} & \quad E_0 = 2.01355 \, c^2 u = 1875.61 \, \text{MeV}
\end{align*}
\]

\[
E_B = m_p c^2 + m_n c^2 - m_d c^2
\]
Deuteron Binding Energy

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Proton \( E_0 = 1.007276 \, c^2u = 938.27 \, \text{MeV} \)

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Deuteron \( E_0 = 2.01355 \, c^2u = 1875.61 \, \text{MeV} \)

\[
E_B = m_p c^2 + m_n c^2 - m_d c^2
\]

\[
E_B = 938.27 + 939.57 - 1875.61 = 2.23 \, \text{MeV}
\]
The Curve of Binding Energy

\[ \frac{B}{A} \text{ (MeV)} \]

- $^2\text{H}$
- $^3\text{He}$
- $^6\text{Li}$
- $^4\text{He}$
- $^{56}\text{Fe}$
- $^{62}\text{Ni}$
- $^{238}\text{U}$
Uranium Disintegration Energy

\[
\frac{236}{92}U \rightarrow \frac{143}{55}Cs + \frac{90}{37}Rb + 3\frac{1}{0}n
\]

Find the Disintegration Energy of Uranium-236 given:

- Uranium – 236 \( E_0 = 236.045563 \ c^2u \)
- Cesium – 143 \( E_0 = 142.927220 \ c^2u \)
- Rubidium – 90 \( E_0 = 89.914811 \ c^2u \)
- Neutron \( E_0 = 1.008665 \ c^2u \)
Uranium Disintegration Energy

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\[ Q = m_Uc^2 - m_{Rb}c^2 - m_{Cs}c^2 - 3m_{n}c^2 \]
Uranium Disintegration Energy

\[ ^{236}_{92}\text{U} \rightarrow ^{143}_{55}\text{Cs} + ^{90}_{37}\text{Rb} + 3^{1}_{0}\text{n} \]

Find the Disintegration Energy of Uranium-236 given:

- Uranium – 236 \( E_0 = 236.045563 \, c^2u \)
- Cesium – 143 \( E_0 = 142.927220 \, c^2u \)
- Rubidium – 90 \( E_0 = 89.914811 \, c^2u \)
- Neutron \( E_0 = 1.008665 \, c^2u \)

\[ Q = m_U c^2 - m_{\text{Rb}} c^2 - m_{\text{Cs}} c^2 - 3m_{\text{n}} c^2 \]

\[ Q = 0.177537 \, c^2u = 165.4 \, \text{MeV} \]
Phenomena made possible by $E_0 = mc^2$

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Momentum
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Binding & Disintegration
Energies
Hydrogen and constituents
Helium and Hydrogen (Deuterium)
Deuteron
Binding Energy
Curve of Binding Energy
Uranium
Disintegration

NASA Photo
Gibbs Peak

Fusion fuels the Sun
Radioactivity heats Earth’s interior

The theory of relativity!
Next Time

Lecture 04: Special & General Relativity

- Energy and momentum
  - Rest energy $E_0 = mc^2$
  - Total energy, and kinetic energy
  - Relativistic momentum
- General relativity