1 Motivation: The symmetry paradigm

Symmetry Paradigm: Over the last century, physics has learned to replace basic assumptions about the dynamics of nature with assumptions about the symmetries of nature. The most familiar examples of this include conservation laws that can be recast as symmetries:

<table>
<thead>
<tr>
<th>Dynamical statement</th>
<th>Symmetry Principle</th>
</tr>
</thead>
<tbody>
<tr>
<td>p conservation</td>
<td>( \leftrightarrow x ) translation invariance</td>
</tr>
<tr>
<td>E conservation</td>
<td>( \leftrightarrow t ) translation invariance</td>
</tr>
<tr>
<td>L conservation</td>
<td>( \leftrightarrow \theta ) translation invariance</td>
</tr>
</tbody>
</table>

Table 1: Conserved Quantities vs. Symmetries of Nature

Conservation laws are a consequence of symmetries.

1.1 Symmetries and Forces

Symmetry is a powerful and predictive organizing principle. It explains and predicts the basic forces (interactions) of nature very economically. Once physicists learned to take symmetry seriously as an organizing principle the fundamental forces of nature were understood as the consequences of symmetries. Forces or interactions arise from symmetries:

<table>
<thead>
<tr>
<th>Dynamics</th>
<th>Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electricity &amp; Magnetism</td>
<td>( \leftrightarrow U(1) ) Gauge Invariance</td>
</tr>
<tr>
<td>Weak Interactions</td>
<td>( \leftrightarrow SU(2) ) Gauge Invariance</td>
</tr>
<tr>
<td>Strong Interactions</td>
<td>( \leftrightarrow SU(3) ) Gauge Invariance</td>
</tr>
<tr>
<td>Gravity (General Relativity)</td>
<td>( \leftrightarrow ) General Covariance</td>
</tr>
</tbody>
</table>

Table 2: Symmetries vs. Forces

A symmetry of nature, or more precisely, of a set of equations which seek to describe nature is a set of transformations which leave the equations invariant. Thus, the study of natures symmetries is the study of transformations of a physical system.

1.2 Unified Symmetries

Grand Unified Theories (GUTS) attempt to unify our understanding of electricity and magnetism, the weak interaction, and the strong interaction. It is possible that these three forces are different manifestations of a single unified force which in turn originate in a unified symmetry

\[ SU(5) \supset SU(3) \times SU(2) \times U(1) \]
An introduction to Groups and Group Algebras

Groups and Reversible Transformations

Reversible transformations of a physical system have a natural interpretation in the language of group theory. Not surprisingly, group theory provides a powerful tool for understanding, classifying and exploiting the symmetry properties of physical systems. To make this connection, let’s formulate the generic properties of reversible transformations of a physical system in abstract mathematical terms. Consider a set of transformations \( a, b, c, \ldots \).

- If \( a \) and \( b \) represent transformation of the system, transforming the system by \( b \) and then by \( a \) can be relabeled as some cumulative transformation \( c \). I will write the composition of two such transformations as \( c = a \cdot b \). The symbol \( c \) means transform by \( b \), then transform by \( a \). This law of composition is often referred to as a law of multiplication.

- If we restrict our transformations to some particular set of transformations, how large should this set be? If we consider a set of transformations \( \{ a, b, c, \ldots \} \), and if we include in this set all transformations which can be formed by combining two other transformations from the set, we say that our set of transformations exhibits closure under the law of composition.

- Making no transformation at all takes the system to itself. So there exits some “identity” transformation, \( e \) which does nothing at all. It follows that \( e \cdot a = a \cdot e = a \).

- Since the transformation is reversible, for every transformation \( a \) there exists an inverse \( a^{-1} \) which undoes the previous transformation. Making a reversible transformation of a physical system and then undoing it is the same as doing nothing at all.

- Finally successive transformation of a physical system are associative, \((a \cdot b) \cdot c = a \cdot (b \cdot c)\). If we consider three successive transformations we are always free to relabel any two successive transformations as a cumulative transformation since at each intermediate stage there is a well defined state of the system.

As it turns out, the properties we have just constructed are those of a well studied class of objects in physics and mathematics called groups. The study of reversible transformation of a physical system is group theory. A group \( G \) is a set of elements \( \{ a, b, c, \ldots \} \) on which a law of composition, \( a \cdot b \), is defined with the following properties:
• **Closure.** \( \forall a, b \in G, a \cdot b \in G. \) Consider all possible transformations of a given type. If you can define a new transformation by combining two other transformations consider it as belonging to the same set of transformations.

• **Identity.** \( \exists \) an identity element \( e \in G, \) such that \( a \cdot e = e \cdot a = a. \) Doing nothing is a transformation of the system. Doing nothing and then doing something is the same as doing something.

• **Inverse.** \( \forall a \in G, \) \( \exists \) an inverse \( a^{-1} \) such that \( a^{-1} \cdot a = a \cdot a^{-1} = e. \) The transformation is reversible. Doing nothing is the same as making a reversible transformation and then undoing it.

• **Associativity.** The composition law is associative: \( \forall a, b \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c). \)

Note that nowhere have we implied that these transformations laws are commutative, \( a \cdot b \) is not in general the same as \( b \cdot a. \) Groups with the property \( a \cdot b = b \cdot a \) are known as Abelian groups.

**Classification of Groups:**

There are a lot of ways to classify groups. The taxonomy that will be most useful for us in this course is:

• **Finite groups:** *Groups with a finite number of elements.*

• **Infinite groups:** *Groups with an infinite number of elements.*
  
  – Infinite discrete groups
  
  – Continuous groups.
    
    * Compact Lie groups
    * Non-compact Lie groups

**Finite groups**

Finite groups include transformations of a physical system which do not depend on continuous parameters. For a finite group, the order of a group is the number of elements in the group. For any finite group, we will list the group composition table as:

**Groups of Order One**

The group of order one contains only the identity element. This is the trivial group which transforms the system to itself.
Table 3: Group Multiplication Table

<table>
<thead>
<tr>
<th></th>
<th>g₁</th>
<th>g₂</th>
<th>...</th>
<th>gn</th>
</tr>
</thead>
<tbody>
<tr>
<td>g₁</td>
<td>g₁ · g₁</td>
<td>g₁ · g₂</td>
<td>...</td>
<td>g₁ · gn</td>
</tr>
<tr>
<td>g₂</td>
<td>g₂ · g₁</td>
<td>g₂ · g₂</td>
<td>...</td>
<td>g₂ · gn</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>gn</td>
<td>gn · g₁</td>
<td>...</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Group Table for the Abstract Group of Order Two

There are several isomorphic realization of finite groups of order two, which include:

- Addition modulo 2 on the integers 0 and 1.
- Multiplication on the elements 1, and −1.
- The Permutations of two objects.

In particle physics, transformations like time reversal \( t \rightarrow -t \), and parity \( \vec{x} \rightarrow -\vec{x} \) can generate groups of order two.

Groups of Order Three

Label the abstract elements of the group as: \( G = \{e, a, b\} \). The compositions \( a \cdot b \) and \( b \cdot a \) must be equal to \( e \), since other possibilities reduce to \( a = e \), or \( b = e \). This is an example of a cyclic group:

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>e</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>e</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>e</td>
<td>a</td>
</tr>
</tbody>
</table>

Table 5: Group Table for Order Three Groups

Realizations of this group

- The cubed roots of unity under multiplication
• Rotations by 0°, 120°, and 240°

The study of finite groups is a big subject, and one that is important in physics, but we will cut our discussion off here and proceed to continuous groups.

Continuous Groups

The Abelian Group $U(1)$

The Abelian group $U(1)$ is composed of unitary $1 \times 1$ matrices. Unitary implies $U^\dagger = U^{-1}$, so that group elements satisfy

$$U^\dagger U = UU^\dagger = 1.$$  

The Hermitian conjugation symbol $\dagger$ means the combined operations of complex conjugation and transpose, which reduces to complex conjugation in this case.\(^1\) Thus, elements of the Abelian group $U(1)$ can be represented by complex phase factors:

$$U(\theta) = e^{i\theta}$$

these group elements are continuous and infinite in number, but the parameter space of the group is finite $0 \leq \theta < 2\pi$. Notice that multiplication by a complex phase satisfies all of the properties of a group. The group $U(1)$ appears in quantum mechanics, because the value of any observables quantity is invariant under redefining the Schrödinger wave function by a complex phase. In other words, quantum mechanics is invariant under the global $U(1)$ transformation (see for example PHYS-215, PHYS-336).

$$\Psi \rightarrow e^{i\theta} \Psi$$

Promotion of this global symmetry to a local symmetry

$$\Psi \rightarrow e^{i\theta(x)} \Psi$$

gives rise to electricity and magnetism. Students interested in exploring this connection further should consult me for additional reading material.

Non-Abelian Continuous Groups

Warming up with $SU(2)$

Before discussing the general properties of continuous groups lets look at the simplest example of a non-Abelian group which depends on a continuous set of parameters. This group is $SU(2)$, it is ubiquitous in particle physics, and a quick survey of its properties will illustrate many of the concepts we will encounter in more general terms later. This group should be familiar to you

\(^1\)The Hermitian conjugate of $U$ is denoted $U^\dagger = (U^T)^* = (U^*)^T$. This notation is ubiquitous in physics, but many math texts use a different notion for hermitian conjugation.
from quantum mechanics since $SU(2)$ has the same algebra as the rotation group $SO(3)$. $SU(2)$ is defined by the group of \textit{special} (determinant one), \textit{unitary} $2 \times 2$ matrices. These matrices satisfy all of the properties of a group. The product of two unitary matrices with determinant one is again a unitary matrix with determinant one, matrix multiplication is associative, $SU(2)$ matrices are invertible (so for each element there is an inverse) and the two by two identity matrix is an element of $SU(2)$.

**Exercise:** By parameter counting, verify that elements of $SU(2)$ are described by three continuous parameters. \textit{Hint:} How many continuous parameters does a general complex two by two matrix have? How many independent constraints does the unitary condition provide? How many constraints does the determinant condition provide?

The \textit{defining} representation of $SU(2)$ is a transformation which acts on an 2-dimensional complex vector:

$$\chi^a \rightarrow \chi'^a = U^a_b \chi^b \quad \text{where} \quad \chi = \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix},$$

(2.2)

where the unitary transformation $U$ is an element of $SU(2)$. We can represent this unitary transformation as $U = \exp(i\theta_A J_A)$. \footnote{Note use of the Einstein summation convention. There is an implied summation on repeated indices: $U^a_b \chi^b \equiv \sum_b U^a_b \chi^b$.} For real $\theta_A$ the $J_A$ are hermitian matrices. \footnote{A hermitian matrix $H$ satisfies $H^\dagger = H$.} The matrices $J_A$ are called the \textit{generators} of $SU(2)$. From the identity $\det e^M = e^{TrM}$, it follows that the $J_A$ are traceless. There are three linearly independent $2 \times 2$ traceless hermitian matrices. The Pauli matrices provide a convenient basis for the generators in the two-dimensional, representation:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(2.3)

Because they act on two dimensional vectors, we say that the generators $J_A = \sigma_A/2$ form a two-dimensional \textit{representation} of the \textit{algebra} of $SU(2)$. This algebra is defined by set of commutation relations:

$$[J_A, J_B] = i\epsilon_{ABC} J_C$$  

(2.4)

where $\epsilon$ is a completely antisymmetric tensor with $\epsilon_{123} = 1$. Exponentiating these generators gives a two dimensional representation of the group elements

$$U(\theta) = e^{i\theta_A \sigma_A/2} = \cos(\theta/2) + i\sigma \cdot n \sin(\theta/2),$$  

(2.5)
where $\theta_A \sigma_A \equiv \theta(n \cdot \sigma)$, and $n$ is a unit vector in the direction of $\vec{\theta}$. For infinitesimal transformations:

$$
\chi^a \rightarrow \chi^a + \frac{i}{2} \theta_A (\sigma_A)^a_b \chi^b.
$$

(2.6)

Although we have defined $SU(2)$ by the action on two-dimensional complex vectors, we can consider the action of the group on higher dimension vectors as well. Consider the tensor which is a product of two two-dimensional representations $T^{ab} = \xi^a \chi^b$. Under $SU(2)$, both of these indices transform simultaneously:

$$
T^{ab} \rightarrow U^a_a U^b_b T^{a'b'}.
$$

(2.7)

This tensor has four independent components, which we can represent it as a four dimensional vector $\zeta$:

$$
T^{ab} = \xi^a \chi^b \quad \text{or} \quad \zeta^a = \begin{pmatrix}
\xi^1 \\
\xi^2 \\
\xi^3 \\
\xi^4
\end{pmatrix}
$$

(2.8)

The transformations in Eqs. 2.6, 2.7 can be used to construct the action of $SU(2)$ in this new basis:

$$
\zeta^a \rightarrow U^a_b \zeta^b.
$$

(2.9)

We could proceed to construct $U$ at this point, but we will not because the tensor $\zeta$ is reducible, it is composed of distinct objects which do not mix under group transformations. This is easiest to see if we study the original tensor $T^{ab}$ which may be decomposed into even and odd parts:

$$
T^{ab} = T^{[ab]} + T^{\{ab\}}, \quad \text{where}
$$

(2.10)

$$
T^{[ab]} = \frac{1}{2} (T^{ab} - T^{ba})
$$

(2.11)

$$
T^{\{ab\}} = \frac{1}{2} (T^{ab} + T^{ba})
$$

(2.12)

Notice that permutations of the indices of $T^{ab}$ commutes with the action of the transformation of Eq. 2.7, i.e. symmetric (anti-symmetric) tensors transform into symmetric (anti-symmetric) tensors. Whenever a representation can be decomposed as a sum of distinct parts which do not mix under group transformations we say that the representation is reducible. The even tensor $T^{(ab)}$ has three distinct elements and is known at the triplet while the odd tensor $T^{\{ab\}}$ has a single element and is known as a singlet. Symbolically we write this as

$$
2 \otimes 2 = 1 \oplus 3
$$

(2.13)

This decomposition suggests a more enlightened choice of basis for $\zeta$. If we redefine $\zeta$ by a non-singular transformation $M$, so that $\zeta$ is decomposed into its
singlet and triplet components, the 4-dimensional $SU(2)$ transformation obtains a block diagonal form by virtue of the fact that these states do not mix:

$$\begin{align*}
\zeta &\rightarrow \zeta' = M\zeta = \begin{pmatrix}
\frac{1}{\sqrt{2}}(\xi^1\chi^2 - \xi^2\chi^1) \\
\frac{1}{\sqrt{2}}(\xi^1\chi^2 + \xi^2\chi^1)
\end{pmatrix} \\
\mathcal{U} &\rightarrow \mathcal{U}' = M\mathcal{U}M^{-1} = \begin{pmatrix}
1 & 0 \\
0 & \mathcal{U}_3
\end{pmatrix},
\end{align*}$$

(2.14)

where $\mathcal{U}_3$ is an element of the three dimensional representation of $SU(2)$, and the singlet state is invariant. This is not the most direct way to construct irreducible representations of $SU(2)$, but it provides a useful illustration of arguments we will generalize later.

**Exercise:** Verify that the antisymmetric tensor $T^{[ab]}$ is invariant under $SU(2)$.

### Invariant Tensors in $SU(2)$

Finally, we note the existence of two special tensors which are $SU(2)$ contains two invariants. The two-dimensional, antisymmetric Levi-Civita tensor is:

$$\epsilon^{ij} = \begin{cases}
1 & i = 1, j = 2 \\
-1 & j = 2, i = 1 \\
0 & i = j
\end{cases}$$

(2.15)

The invariance of this tensor follows from the antisymmetry of $\epsilon$, and the requirement that elements of $SU(2)$ satisfy $\det \mathbf{U} = 1$.

$$\epsilon^{ac} \rightarrow U^a_b U^c_d \epsilon^{bd} = \epsilon^{ac}$$

The Kronicker delta tensor

$$\delta^{ij} = \begin{cases}
1 & i = j \\
0 & i \neq j
\end{cases}$$

(2.16)

is invariant under $SU(2)$ transformations.

### Ladder Operators in $SU(2)$

The standard method for constructing irreducible representations of $SU(2)$ involves the use of raising and lowering operators. Many of You will have seen this in previously quantum mechanics. $SU(2)$ has a Casimir operator

$$\mathbf{J}^2 = \mathbf{J}_1^2 + \mathbf{J}_2^2 + \mathbf{J}_3^2$$

(2.17)

which commutes with all of the generators of $SU(2)$:

$$[\mathbf{J}^2, \mathbf{J}_i] = 0.$$
From the generators, we can construct a pair of raising and lowering operators (sometimes known as ladder operators) $J_{\pm}$

$$J_{\pm} = J_1 \pm iJ_2$$  \hspace{1cm} (2.18)

You are encouraged to verify that these new operators satisfy the commutation relations:

$$[J_3, J_{\pm}] = \pm J_{\pm}$$  \hspace{1cm} (2.19)

$$[J_+, J_-] = 2J_3$$  \hspace{1cm} (2.20)

Because $J^2$ and $J_3$ commute, we can simultaneously diagonalize $J^2$ and $J_3$. Let $| \lambda, m \rangle$ denote an eigenvector of $J^2$ and $J_3$ with eigenvalues $\lambda$ and $m$ respectively.

$$J^2 \left| \lambda, m \right\rangle = \lambda \left| \lambda, m \right\rangle$$  \hspace{1cm} (2.21)

$$J_3 \left| \lambda, m \right\rangle = m \left| \lambda, m \right\rangle$$  \hspace{1cm} (2.22)

If this notation is unfamiliar to you, you can think of $| \lambda, m \rangle$ as a column vector. Consider two new state of the system, i.e. two new column vectors which are constructed by acting on the eigenvectors above with the raising and lowering operators

$$| x_{\pm} \rangle = J_{\pm} \left| \lambda, m \right\rangle$$

. We will demonstrate that:

- These new states (column vectors) are still eigenstates (eigenvectors) of $J^2$ and $J_3$.
- These new states have the same eigenvalue under $J^2$.
- The eigenvalue of $J_3$ changes by one unit.

Because $[J^2, J_{\pm}] = 0$, the raising and lowering operators, do not change the eigenvalue of $J^2$.

$$J^2 \left| x_{\pm} \right\rangle = J^2 \left| J = J_\pm J_\mp \left| \lambda, m \right\rangle = J_{\pm} \lambda \left| \lambda, m \right\rangle = \lambda \left| x_{\pm} \right\rangle$$  \hspace{1cm} (2.23)

Having shown that these new states are eigenstates of $J^2$, and that the eigenvalue of $J^2$ remains unchanged, we relabel these new states to reflect this fact.

$$| x_{\pm} \rangle \longrightarrow | \lambda, x_{\pm} \rangle = J_{\pm} \left| \lambda, m \right\rangle$$

By contrast, the raising and lowering operators change the eigenvalue of $J_3$ by one unit.

$$J_3 \left| \lambda, x_{\pm} \right\rangle = J_3 \left| J = J_\pm J_\mp \left| \lambda, m \right\rangle = (J_\pm J_3 + [J_3, J_{\pm}]) \left| \lambda, m \right\rangle = (J_{\pm} J_\mp \left| \lambda, m \right\rangle = (m \pm 1) \left| \lambda, x_{\pm} \right\rangle$$  \hspace{1cm} (2.24)

With this result, we identify $x_{\pm} = (m \pm 1)$, and write the raising and lowering operations as

$$J_{\pm} \left| \lambda, m \right\rangle = \lambda \left| m \pm 1 \right\rangle.$$
Technically the equality above should really be a proportionality. We have not yet specified how we are normalizing the eigenstates. If we multiply any of these eigenstates by a number the eigenvalues do not change.

We can act on these new states with raising and lowering operators to increment or decrement the eigenvalue of $m$ again. Continuing this process, the raising and lowering operators can be used to generate a series of eigenvectors:

$$| \lambda, m_{\text{min}} \rangle, \ldots, | \lambda, m - 2 \rangle, | \lambda, m - 1 \rangle, | \lambda, m \rangle, | \lambda, m + 1 \rangle, | \lambda, m + 2 \rangle, \ldots, | \lambda, m_{\text{max}} \rangle$$

This series must terminate since

$$J^2 \geq J^2_3, \quad \Rightarrow \quad \lambda \geq m_{\text{max}}^2, m_{\text{min}}^2$$

For the series to terminate we must have

$$J_+ | \lambda, m_{\text{max}} \rangle = J_- | \lambda, m_{\text{min}} \rangle = 0$$

The termination of this series relates $\lambda$ to $m_{\text{max}}$ and $m_{\text{min}}$. Using the identity,

$$J^2 = J_+ J_- + J^2_3 + J_3$$

a relation from $\lambda$ and $m_{\text{max}}$ follows from

$$J^2 | \lambda, m_{\text{max}} \rangle = (J_+ J_+ + J^2_3 + J_3) | \lambda, m \rangle = (m_{\text{max}}^2 + m_{\text{max}}) | \lambda, m \rangle = m_{\text{max}} (m_{\text{max}} + 1) | \lambda, m \rangle = \lambda | \lambda, m \rangle$$

Thus $\lambda = m_{\text{max}} (m_{\text{max}} + 1)$ A similar analysis for the lowering operator on $| \lambda, m_{\text{min}} \rangle$ yields $\lambda = m_{\text{min}} (m_{\text{min}} - 1)$. Since $m_{\text{max}} \geq m_{\text{min}}$, it follows that $m_{\text{max}} = -m_{\text{min}}$. Relabeling the maximum value of $m$ by $j$, with a slight shift in notation, the eigenvalue equations become

$$J^2 | j, m \rangle = j(j + 1) | j, m \rangle$$

$$J_3 | j, m \rangle = m | j, m \rangle$$

Note that for fixed $j$ there are $d = 2j + 1$ possible choices for $m$: $m = -j, -j + 1, \ldots, j - 1, j$, where $d$ is the dimension of the representation. Note that the values of $j$ are either integer, or half integer $j = n + \frac{1}{2}$. To connect this notation back with the defining representation of SU(2)

$$\begin{pmatrix} \frac{1}{2}, \frac{1}{2} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2}, -\frac{1}{2} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The states conjugate to these are

$$\begin{pmatrix} \frac{1}{2}, \frac{1}{2} \end{pmatrix} = (10)^\dagger, \quad \begin{pmatrix} \frac{1}{2}, -\frac{1}{2} \end{pmatrix} = (01)^\dagger$$
These states have been ortho-normalized:

\[
1 = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right\rangle,
\]

\[
0 = \left\langle \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right\rangle.
\]

In general we choose to orthonormalize our states so that

\[
\langle j, m | j, m' \rangle = \delta_{mm'} \quad (2.33)
\]

The commutation relations above imply that raising and lowering operators change the value of \( m \) by one unit while leaving \( j \) unchanged, but the normalization of a state obtained by raising or lowering may change, so we write

\[
\mathbf{J}_\pm | j, m \rangle = C_\pm(j, m) | j, m \pm 1 \rangle \quad (2.34)
\]

\[
\langle j, m | \mathbf{J}_\mp = C_{\mp}(j, m) \langle j, m \pm 1 | \quad (2.35)
\]

from which we obtain

\[
| C_\pm |^2 = \langle j, m | \mathbf{J}_\pm \mathbf{J}_\mp | j, m \rangle
\]

\[
= \langle j, m | \mathbf{J}^2 - \mathbf{J}^2_3 \pm \mathbf{J}_3 | j, m \rangle
\]

\[
= j(j+1) - m^2 \mp m \quad (2.36)
\]

So that

\[
\mathbf{J}_\pm | j, m \rangle = \eta_\pm(j, m) \sqrt{j(j+1) - m(m \pm 1)} | j, m \pm 1 \rangle \quad (2.37)
\]

where \( \eta_\pm(j, m) \) is unimodular phase which satisfies \( \eta_+(j, m) = \eta_-(j, m + 1) \) due to the fact that \( \mathbf{J}_- = \mathbf{J}_+^T \). The conventional choice \( \eta_+ = \eta_- = 1 \). From the relations above it is easy to work out the generators in any representation. In the basis where

\[
| 1, 1 \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad | 1, 0 \rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad | 1, -1 \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\]

the raising and lowering operators in the adjoint representation are

\[
\mathbf{J}_+ = \sqrt{2} \begin{pmatrix} 0 & \eta_1 & 0 \\ 0 & 0 & \eta_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{J}_- = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \end{pmatrix}
\]

\[
(2.39)
\]

where \( \eta_1 \) and \( \eta_2 \) are unimodular phase factors which are 1 in the standard phase convention. The generators follow from \( \mathbf{J}_1 = \frac{1}{2} (\mathbf{J}_+ + \mathbf{J}_-) \), \( \mathbf{J}_2 = \frac{-i}{2} (\mathbf{J}_+ - \mathbf{J}_-) \), and the eigenvalue equations.

\[
\mathbf{J}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \eta_1 & 0 \\ \eta_1 & 0 & \eta_2 \\ 0 & \eta_2 & 0 \end{pmatrix}, \quad \mathbf{J}_2 = \frac{-i}{\sqrt{2}} \begin{pmatrix} 0 & -\eta_1 & 0 \\ \eta_1 & 0 & \eta_2 \\ 0 & -\eta_2 & 0 \end{pmatrix}, \quad \mathbf{J}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

\[
(2.40)
\]
Exercise: Verify the constants $C_{\pm}(j, m)$ below for the raising and lowering operations on the states with $j = 1$:

\[
\begin{align*}
\mathbf{J}_+ |1, 1\rangle &= 0 & \mathbf{J}_- |1, 1\rangle &= \sqrt{2} |1, 0\rangle \\
\mathbf{J}_+ |1, 0\rangle &= \sqrt{2} |1, 1\rangle & \mathbf{J}_- |1, 0\rangle &= \sqrt{2} |1, -1\rangle \\
\mathbf{J}_+ |1, -1\rangle &= \sqrt{2} |1, 0\rangle & \mathbf{J}_- |1, -1\rangle &= 0 \\
\end{align*}
\]

Note we have chosen a convention where the phase factors $\eta_{\pm}$ have been set to one.

Exercise: Using the relations above, construct the ladder operators of Eq. 2.39.

Using these raising and lowering operators and the eigenvalue equation for $\mathbf{J}_3$ verify that the generators for the three dimensional representation of $SU(2)$ are given by Eq. 2.40.

Exercise: Verify that these matrices satisfy the algebra of $SU(2)$.

### Lie Groups and Lie Algebras

Groups in which the group elements are labeled by a set of continuous parameters, with a multiplication law that depends smoothly on those parameters are known as Lie groups. These groups are of particular interest to us because field theories contain a variety of continuous symmetries. The continuous internal symmetries of quantum field theory (those not connected with space-time indices) are compact Lie groups. I will not discuss these distinction between compact and non-compact Lie groups except to say that the parameters of a compact Lie group vary over a closed interval. Elements of a compact Lie group can always be represented by unitary operators. Non-compact Lie groups are also relevant in particle physics, and an examples of non-compact Lie groups include the proper Lorentz and Poincare groups.

For a transformation of the physical system labeled by a continuous set of real parameters $\xi$ whose elements (transformations) can be taken arbitrarily close to the identity, \(^5\)

\[ U(\xi) = \exp i\xi^A T^A = I + i\xi^A T^A + \ldots, \tag{2.42} \]

where the $T^A$ are generators for the group $G$. For unitary $U(\xi)$, the $T^A$ are a set of linearly dependent hermitian operators. \(^6\)

The generators of a group are especially easy to work with because they form a vector space. We can add generators together to form other generators, and we can multiply them by numbers. The commutation relations of the generators

\(^5\)Strictly speaking Lie group can contain elements which can not be obtained from the identity by continuous changes in the parameters $\xi$. However, the group members which can be deformed arbitrarily close to the identity are sufficient to determine the algebra of the group.

\(^6\)Group representations can be constructed from non-unitary operators as long as they satisfy the properties of a group. However, every representation of a compact Lie group is equivalent to a representation by unitary operators.
of a Lie group are called the Lie algebra of the group. Defined mathematically, a Lie algebra is a vector space \( L \) over a field \( F \), together with a bilinear operation \([x, y] = xy - yx\), which satisfies the following properties:

- \([x + y, z] = [x, z] + [y, z]\), for \( x, y, \) and \( z \) in \( L \)
- \([ax, y] = a[x, y]\), for \( x, y, \) in \( L \), and \( a \) in \( F \)
- \([x, y] = -[y, x]\) for \( x, y, \) in \( L \)
- \([x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\), for \( x, y, z \) in \( L \).

A field \( F \) is a set on which the standard arithmetic properties governing addition, subtraction, multiplication, and division hold. We will always take the field \( F \) to be the field of real numbers \( R \).

The Lie algebra is:

\[ [T^A, T^B] = i f^{AB}_C T^C, \tag{2.43} \]

where the \( f^{AB}_C \) are called the structure constants of the group. 7 For Hermitian \( T^A \), the structure constants are a set of real numbers. The structure constants are completely determined by the group multiplication law, \( U(x)U(y) = U(x \cdot y) = U(z) \):

\[ e^{ix_A T^A} e^{iy_B T^B} = e^{iz_C T^C} \tag{2.44} \]

\( z_A \) can be related to \( x_A \) and \( y_A \) to any order by expanding the group elements above. With a little algebra we find:

\[ z_A = x_A + y_A - \frac{1}{2} f^{BC}_A x_B y_C + \ldots \tag{2.45} \]

**Representations**

The features of representation discussed previously for \( SU(2) \) generalize to general Lie Algebras: For an abstract group element labeled by \( N \) continuous parameters \( (\theta_1, \theta_1, \ldots, \theta_N) \), a representation is a specific realization of the group elements and group generators by matrices. The dimension of a representation is the dimension of the vector space on which the matrices act. For most purposes, we will make use of representations of group elements \( U(\theta) \) as opposed to the abstract transformations \( \theta \). A set of operators \( U(\theta) \) is a representation of the group if

- For every transformation \( \theta \) in \( G \), there is a unitary operator \( U(\theta) \).
- The mapping of \( \theta \) to \( U(\theta) \) preserves the law of composition: \( U(\theta)U(\phi) = U(\theta \cdot \phi) \). This implies that the algebra of the group generators is the same for every representation of the groups.

7 For hermitian generators with a common normalization we do not need to distinguish between upper and lower indices on the structure constants.
A representation of the group $U$ is reducible if it is equivalent to a representation $U'$ with a block diagonal form:

$$U' = M U M^{-1} = \begin{pmatrix} U_1' & 0 & 0 \\ 0 & U_2' & 0 \\ 0 & 0 & \ddots \end{pmatrix},$$

where $M$ is a non-singular matrix.

If $U(\theta)$ forms a representation of the group, then $U^*$ forms another representation of the group known as the complex conjugate representation. A representation is called real if it is equivalent to its complex conjugate representation. Denote the generators in a particular representation $R$ by $T^A(R)$. If there exists $M$ such that

$$MT^A(R)M^{-1} = -T^A^*(R) \quad \text{for all } A$$

then $R$ is a real representation, otherwise $R$ is known as a complex representation. Real representations can be further subdivided into real-positive (or just plain real) and real-negative (or pseudo-real) depending on whether $M$ is symmetric or anti-symmetric respectively.

**Exercise:** Show that the two dimensional representation of $SU(2)$ is pseudo-real.

**Exercise:** Show that the adjoint representation of $SU(2)$ is real.

**The Adjoint Representation**

In a specific representation the generators of a group, $T^A$, are $n \times n$ matrices where $n$ is the dimension of the representation, and $A = 1, 2, \ldots, N$ where $N$, as we shall see below, is the dimension of the adjoint representation. Although we can always take the $T^A$ as hermitian matrices, sometimes it is useful to work in a non-hermitian basis. These generators satisfy the Jacobi identity.

$$[X^A, [X^B, X^C]] + [X^C, [X^A, X^B]] + [X^B, [X^C, X^A]] = 0$$

Written in terms of the structure constants this becomes

$$0 = f^{AB} f^{CD} E + f^{BC} f^{AD} E + f^{CA} f^{BD} E$$

We can define a set of matrices from the structure constants

$$(F^A)^B = -if^{AB} C.$$
These matrices themselves form a representation of the group, the adjoint representation:

\[ [F^A, F^B] = i f^{AB}_C F^C. \]  

(2.51)

This is true for any basis, the generators \( F \) need not be hermitian. \(^9\)

The dimension of the adjoint representation is just the number of real parameters needed to describe a group element. The structure constants depend on the basis we choose for our generators \( T^A \). The adjoint representation provides a fairly convenient representation to look for a basis which simplifies the form of the structure constants. \( Tr(X^A X^B) \) is a real symmetric matrix, so we can redefine our adjoint generators to diagonalize this matrix

\[ Tr X^A X^B = k_A \delta^{AB}. \]  

(2.52)

We can further re-scale the generators, and re-scale \( k_A \), but notice that we can not change the sign of \( k_A \). For compact semi-simple Lie algebras we can take all of the \( k_A \)’s as positive, and henceforth we will implicitly normalize these generators so that

\[ Tr X^A X^B = k_D \delta^{AB}, \]  

(2.53)

where \( k_D \) is a representation dependent constant. Each of these orthogonal generators can be associated with a state of the adjoint representation.

\[ X^A \leftrightarrow | X^A \rangle \equiv | A \rangle \quad \text{where} \quad \langle A | B \rangle = tr(X^A X^B)/k_D = \delta^{AB} \]  

(2.54)

For the adjoint representation, in any orthonormal basis where \( (X^A)^B_C = -i f^{AB}_C \), the action of the group generators on a state in the adjoint representation is

\[ X^A | X^B \rangle = -i f^{AC}_B | X^C \rangle \quad \text{for any orthonormal} \ X \]
\[ = i f^{AB}_C | X^C \rangle = | [X^A, X^B] \rangle \quad \text{for hermitian} \ X \]  

(2.55)

To see this note that if we define matrices \( (X^A)^B_C = \langle A | X^A | C \rangle \), and insert an orthonormal set of states \( 1 = \sum_D | D \rangle \langle D | \):

\[ X^A | X^B \rangle = \sum_D | X^D \rangle \langle X^D | X^A | X^B \rangle \]
\[ = \sum_D | X^D \rangle (X^A)^D_B = -i f^{AD}_B | X^D \rangle \]
\[ = | [X^A, X^B] \rangle \quad \text{for hermitian} \ X \]  

(2.56)

The last line follows from the fact that the structure constants are totally antisymmetric in this basis if the generators are hermitian:

\[ -i f^{AB}_C = Tr[X^A, X^B] X^C = Tr[X^B, X^C] X^A = Tr[X^C, X^A] X^B \]  

(2.57)

\(^9\)This is not the same basis that we constructed previously for the adjoint representation of \( SU(2) \), because our eigenvectors are not identical to Eq. 2.38.
Casimir and Index

Next we introduce two invariants, the index of a representation, and the quadratic Casimir. For every representation $R$:

$$[T^A(R), T^B(R)] = i f^A_{\; BC} T^C(R),$$  \hfill (2.58)

where $T^A(R)$ is a representation of a group generator. In often used notation, the adjoint representation is denoted by $G$ and the fundamental representation is denoted $F$. The quadratic Casimir of a representation $R$ is $C_2(R)$:

$$T^A(R)T^A(R) = C_2(R) I(R),$$  \hfill (2.59)

where there is an implied sum on $A$, and $I(R)$ is the $d(R) \times d(R)$ dimensional identity matrix, where $d(R)$ is the dimension of representation $R$. For the adjoint representation,

$$f^A_{\; BC} f^B_{\; DC} = C_2(G) \delta^{AB}$$  \hfill (2.60)

A second invariant is called the index of a representation $C(R)$

$$trT^{A\dagger}(R)T^B(R) = C(R) \delta^{AB}$$  \hfill (2.61)

Equating the trace of Eq. 2.59 and the sum of Eq. 2.61, it follows that

$$d(R)C_2(R) = d(G)C(R),$$  \hfill (2.62)

Note that $C_2(G) = C(G)$. For any representation $R$

$$f^A_{\; BC} = \frac{-i}{C(R)} tr \left\{ [T^A, T^B] T^{C\dagger} \right\}$$  \hfill (2.63)

**Exercise:** Using the invariants above, verify that for any unitary redefinition of the generators $X^A(R) = U(R)^A_{\; \; B} T^B$, the new basis for the generators has the same invariants

$$X^{A\dagger}(R)X^A(R) = C_2(R) I(R)$$
$$trX^{A\dagger}(R)X^B(R) = C(R) \delta^{AB}$$  \hfill (2.64)

**Exercise:** Consider a non-singular, but non-unitary redefinition of the generators which does not preserve the normalization of the generators: $X^A(R)' = M(R)^A_{\; \; B} T^B$. Define a contravariant generators with a compensating normalization, $X_A = X^{A\dagger}/k_A$, where $TrX^AX^{A\dagger} = k_A C(R)$ (no sum on A). Show that $X'^A(R) = X_b (M^{-1})^B_{\; \; A}$. Writing the algebra as:

$$[X^A, X^B] = i f^A_{\; BC} X^C$$
$$[X_A, X_B] = -i f^A_{\; BC} X_C$$  \hfill (2.65)

show that the Casimir is given by:

$$C(G)C_2(R) I(R) = f^A_{\; BC} f^{EB}_{\; \; CD} X^A(R) X^E(R)$$
$$= f^A_{\; BC} f^{EB}_{\; \; CD} X'^A(R) X'_E(R)$$  \hfill (2.66)

where the metric tensor $G^E_A = f^A_{\; BC} f^{EB}_{\; \; CD}$
A few useful results:

\[
d(G) = \begin{cases} 
N^2 - 1 & SU(N) \\
N(N - 1)/2 & SO(N) \\
N(N + 1)/2 & Sp(N)
\end{cases}
\]

(2.67)

A few useful results for \(SU(N)\):

\[
C(N) = \frac{1}{2}, \quad C_2(N) = \frac{N^2 - 1}{2N}, \quad C_2(G) = C(G) = N
\]

(2.68)

**The Orthogonal group \(O(n)\)**

The \(n\)-dimensional orthogonal group \(O(n)\) is defined as the set of transformations on a real vector \(\phi_1, \ldots, \phi_n\) which preserves the product \(\phi \cdot \phi\). These transformations can be thought of as rotations and reflections of a rigid coordinate system in \(n\) spatial dimensions. For transformations of \(\phi\) of the form:

\[
\phi \to \phi' = R\phi
\]

(2.69)

the inner product \(\phi \cdot \phi\) is preserved by any real orthogonal matrix \(R\):

\[
R^T R = I.
\]

(2.70)

With \(n(n + 1)/2\) conditions on \(n^2\) parameters, there are \(N = n(n - 1)/2\) free parameters. Thus, \(O(n)\) transformations are labeled by \(N = n(n - 1)/2\) continuous parameters which we may take as a generalized set of Euler angles \(\theta_1, \ldots, \theta_N\), and by possible discrete reflections. Because orthogonal matrices have the property that \(\det R = \pm 1\) i.e. there are two distinct classes of orthogonal matrices: pure rotation (\(\det R = 1\)), and rotation and reflection (\(\det R = -1\)). \(O(n)\) is an example of a group which, in addition to continuous parameters, requires a discrete label, indicating the existence of a reflection, to characterize all of its elements elements. Such groups are said to be mixed continuous groups. The sub-group consisting of pure rotations, \(\det R = 1\), is known as \(SO(n)\).

**SO(3): The Proper Rotation Group**

\(SO(3)\) is the subgroup of \(O(3)\) consisting of proper rotations.\(^{10}\) The designation \(SO\) refers to special (aka determinant one) and orthogonal. Elements of \(SO(3)\) can be written as:

\[
R = e^{i\theta_n J_n}
\]

(2.71)

\(^{10}\) Proper rotations are those which are pure rotations. They do not contain spatial reflections.

\(^{11}\) This group \(SO(3)\) is sometimes called \(O_3\) in some texts, and should not to be confused with \(O(3)\).
where $n = 1, 2, 3$. Because $R$ is real and unitary, the generators $J$ are $3 \times 3$ imaginary, hermitian and thus anti-symmetric matrices. There are three linearly-independent imaginary, $3 \times 3$ Hermitian matrices. A standard basis for the three generators of $SO(3)$ is:

$$
J_x = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_y = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_z = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

The algebra of these generators is identical to the algebra of $SU(2)$:

$$[J_i, J_j] = i \epsilon_{ijk} J_k. \quad (2.73)$$

It follows that these matrices provide an equally valid representation of the generators for the three dimensional representation of $SU(2)$.

**Spinor Representation**

Since the Algebra of $SO(3)$ is isomorphic to $SU(2)$, we can construct a two dimensional representation of $SO(3)$. This representation is known as the spinor representation. As in $SU(2)$, the spinor representation of the group acts on a complex two dimensional vector, and the generators can be chosen as the Pauli matrices mod two. Although $SU(2)$ and $SO(3)$ share the same algebra, the groups are not identical. The defining two-dimensional representation of $SU(2)$ is a faithful representation, while the two-dimensional spinor representation of $SO(3)$ is double-valued. Consider the $SO(3)$ transformation:

$$U(0, 0, \theta) = e^{i\theta J^3} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.74)$$

For this transformation we identify $U(0, 0, 0) = U(0, 0, 2\pi) = I$, while for spinor representation,

$$U_{1/2}(\theta) = e^{i\theta \sigma_3/2} = \cos(\theta/2)I + i\sigma_3 \sin(\theta/2), \quad (2.75)$$

which implies $U(0, 0, 0) = I$ and $U(0, 0, 2\pi) = -I$. Thus there is a two to one correspondence between $U_{1/2}(\theta_1, \theta_2, \theta_3)$ and the abstract transformation $\theta_1, \theta_2, \theta_3$. For the specific case above:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad -1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

This double-valuedness is, however, allowable in quantum theories since the phase of a state has no physical meaning. For an abstract group transformation $\theta$ and a state $|\psi\rangle$, $U(\theta)|\psi\rangle$, and $\eta U(\theta)|\psi\rangle$, represent the same state, for some complex phase $\eta$. \quad (2.76)
Tensor methods in $SU(n)$

The defining representation for $SU(n)$ is the group of all unitary $n \times n$ complex matrices with determinant one: $U^\dagger U = UU^\dagger = I$, and $\det U = 1$. This latter requirement, $\det U = e^{i\xi^A T^A} = 1$, implies $Tr T^A = 0$. Since $T^A$ is both hermitian and traceless we know that the dimension of the adjoint representation is $n^2 - 1$. Hence the action of a group element is specified by $n^2 - 1$ real parameters $\xi^A$

$$U(\xi) = \exp i\xi^A T^A. \quad (2.77)$$

The fundamental (defining) representation of the group acts on an $n$-dimensional complex vector:

$$i! \rightarrow \zeta^i = U^i_j \zeta^j \quad (2.78)$$

The representation conjugate to $n$ transforms as: $\eta_i \sim \eta^i$. The transformation for the conjugate representation $\bar{n}$ is then:

$$\eta_i \rightarrow \eta'_i = U^i_j \eta_i \quad (2.79)$$

Higher rank tensors transform analogously

$$T^L_{k_1 k_2 ... k_q} = U^{i_1}_{k_1} U^{i_2}_{j_2} ... U^{i_p}_{j_p} U^i_{k_1} U^{i_2}_{k_2} ... U^i_{k_q} T^L_{i_1 j_1 ... j_p} \quad (2.80)$$

With this notation, it is manifest that we can form group singlets by contracting complete sets of upper and lower indices. $SU(n)$ has two invariants tensors: $\delta^i_j$ and $\epsilon_{i_1 i_2 ... i_N}$. The invariance of $\delta$ is a consequence of the unitarity of $U$, and the invariance of $\epsilon$ follows from $\det U = 1$ The Levi-Civita tensor is a totally anti-symmetric tensor which satisfies:

$$\epsilon_{i_1 i_2 ... i_N} = \epsilon_{i_1 i_2 i_3 ... i_N} = \begin{cases} 
+1 & \text{if } (i_1, i_2, ... , i_N) \text{ is an even permutation of } (1, ..., n) \\
-1 & \text{if } (i_1, i_2, ... , i_N) \text{ is an odd permutation} \\
0 & \text{otherwise}
\end{cases} \quad (2.81)$$

Since the Levi-Civita is and invariant, we can use it to raise and lower tensor indices.

$$T^L_{i_1 i_2 ... i_N} = \epsilon_{i_1 i_2 ... i_N} \zeta^i \quad (2.82)$$

By taking successive direct products of the fundamental representation $n$, we can construct any other representation of $SU(n)$. A general product of two fundamental representations is not in general irreducible. The fact that any permutation symmetry of the tensor indices in Eq. 2.80 is preserved by the action of $SU(n)$ suggests a prescription for finding irreducible representations of $SU(n)$ using tensor methods is:

- Construct tensors with a given number of upper and lower indices
- Divide them into as many invariant parts as possible by classifying components according to symmetry properties of the upper and lower indices. Since permutations of upper (lower) indices commute with the group transformation, tensors separated by their permutation symmetry will not mix under group transformations.
• Eliminate the pieces which can be shown to be equivalent to tensors of a lower rank. This can be done by contracting pairs of upper and lower indices, or by contracting indices with the epsilon tensor.

• Identify the remaining parts with irreducible representations

This procedure can be used to fully reduce a product representation of $SU(n)$. For example consider the tensors which lies in the direct product of the fundamental representation of $SU(n)$, $T^{ij} = x^i y^j$. There are $n^2$ independent elements which can be written as:

\[
T^{ij} = T^{[ij]} + T^{(ij)}, \text{ where}
\]

\[
T^{[ij]} = \frac{1}{2} \{ x^i y^j - x^j y^i \}, \text{ and}
\]

\[
T^{(ij)} = \frac{1}{2} \{ x^i y^j + x^j y^i \}, \text{ or}
\]

\[
n \otimes n = \frac{n(n-1)}{2} \oplus \frac{n(n+1)}{2} \tag{2.83}
\]

The symbols $\oplus$ and $\otimes$ denote direct products and direct sums respectively. Note that in the case of $SU(3)$, we can identify $\frac{3(3-1)}{2} = 3$, with the epsilon tensor

\[
3 = \epsilon_{ijk} T^{[jk]} \tag{2.84}
\]

Next consider tensors which lie in the direct product of the fundamental representation and its conjugate, $T^j_j = x^i y_j$. Because the upper and lower indices transform differently under the gauge group we can no longer use symmetrization as a tool. However, we still have the ability to contract upper and lower indices. For this product there are $n^2$ independent elements which can be written as the sum of a singlet and the adjoint:

\[
n \otimes \bar{n} = n^2 - 1 \oplus 1
\]

\[
T^i_j = A^i_j + \frac{1}{n} S \delta^i_j \text{ where}
\]

\[
A^i_j = x^i y_j - \frac{1}{n} x^k y_k \delta^i_j, \text{ the adjoint} \tag{2.85}
\]

**The Group $SU(3)$**

$SU(3)$ has eight generators. For the defining representation $3$, the physicists standard basis for the generators of $SU(3)$ are the Gell-Mann matrices $T^A = \lambda^A/2$: 
\[ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \]

In this basis, the generators satisfy:

\[ \text{tr}(T^A T^B) = \frac{1}{2} \delta_{AB} \]

\[ [T^A, T^B] = i f^{AB}_{\phantom{AB}C} T^C \]

where \( f^{AB}_{\phantom{AB}C} \) is a totally antisymmetric tensor with \( f_{123} = 1, f_{147} = f_{246} = f_{257} = f_{345} = f_{356} = f_{367} = 1/2, f_{458} = f_{679} = \sqrt{3}/2, \)

**SU(2) subgroups of SU(3)**

**Sub-algebra** Suppose \( L \) is a Lie algebra, \( M \) is a sub-algebra of \( L \) if \([x, y] \in M \forall x, y \in M \) or zero under commutation relations.

We can construct three convenient SU(2) subgroups of SU(3) which satisfy the angular momentum commutation relations:

\[ \left[ \lambda^\alpha \frac{a}{2}, \lambda^\beta \frac{a}{2} \right] = i \epsilon_{\alpha \beta \gamma} \lambda^\gamma \frac{a}{2} \]

In terms of the Gell-Man matrices, these subgroups are:

- **I-spin:** \( \{I_1, I_2, I_3\} = \{T_1, T_2, T_3\} \)
- **V-spin:** \( \{V_1, V_2, V_3\} = \{T_4, T_5, \frac{1}{2} T_3 + \frac{\sqrt{2}}{2} T_8\} \)
- **U-spin:** \( \{U_1, U_2, U_3\} = \{T_6, T_7, \frac{1}{2} T_3 + \frac{\sqrt{3}}{2} T_8\} \)

We can define raising and lowering operators for each of the SU(2) subgroups: \( J_\pm = J_1 \pm i J_2 \), where \( J = \{I, V, U\} \). These subgroups, useful in discussing both the quark model and chiral symmetry, have the following properties:

**I-spin:** SU(2)_I operators commute with \( Y = \frac{2}{\sqrt{3}} T^8 \)

**U-spin:** SU(2)_U operators commute with \( Q = T^3 + \frac{1}{\sqrt{3}} T^8 \). When later consider approximate SU(2) and SU(3) flavor symmetries it will be helpful to note that: as consequence of this, particles in the same U-spin representation have the same electromagnetic properties.

**V-spin:** SU(2)_V operators commute with \( Q = T^3 - \frac{1}{\sqrt{3}} T^8 \).
2.0.1 The $SU(3)$ Octet

\[ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \]

\[
(1, 1) \\
(-1, 2) \quad (2, -1) \\
(0, 0)_2 \quad (0, 0)_1 \\
(-2, 1) \quad (1, -2) \\
(-1, -1)
\]

In this example there is a natural basis for degenerate weights

\[
E_{-\alpha_i} \mid w_1, w_2 \rangle = N_{-\alpha_i, w} \mid w_1 - A_{i_1}, w_2 - A_{i_2} \rangle
\]

The standard recursion relation above is incomplete.

\[
E_{-\alpha} \mid 0, 0 \rangle_2 = \frac{1}{\sqrt{2}} \mid 1, -2 \rangle. \text{ Degenerate weights are not orthogonal. } \langle 00 \mid 00 \rangle_2 = \frac{1}{2}
\]

\[
N_{-\alpha_i, (w)} = + \left[ w_i + N_{-\alpha_i, w} \langle (w) \mid (w) \rangle \right]^{1/2}
\]

No unique basis for degenerate weights.

2.0.2 The 27 of $SU(3)$
Table 6: The 27 dimensional representation of $SU(3)$. 

\[
\begin{array}{cccc}
(22) & (30) & (03) \\
(4-2) & (11)_{1} & (11)_{2} & (-24) \\
(2-1)_{\alpha} & (2-1)_{\beta} & (-12)_{\alpha} & (-12)_{\beta} \\
(3-3) & (00)_{x} & (00)_{y} & (00)_{z} & (-33) \\
(1-2)_{\gamma} & (1-2)_{\delta} & (-21)_{\epsilon} & (-21)_{d} \\
(2-4) & (-1-1) & (-1-1) & (-42) \\
(0-3) & (-30) \\
(-2-2)
\end{array}
\]