The Ising Model

Mathematical Sciences Seminar

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Image from: http://www.wikipedia.org
The Lenz-Ising Model

Interaction proposed by Lenz in the 1920’s.

\[ E = -\epsilon \]

or

\[ E = +\epsilon \]

For a system of dipoles

\[ U = -\epsilon \sum_{pairs} s_i s_j, \quad s_i = \pm 1 \]
Consider a particular quantum state \( n \) with energy \( E_n \). The probability of finding the system in state \( n \):

\[
P(n) = \frac{1}{Z} e^{-\beta E_n}, \quad \beta = 1/kT
\]

Partition function:

\[
Z = \sum_{n=0}^{\infty} e^{-\beta E_n} = e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3} + \ldots
\]

Average Energy:

\[
\bar{U} = -\frac{\partial}{\partial \beta} \ln Z = \sum_n E_n P(n)
\]
Consider a one dimensional array of $N$-spins...

\[ i = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \cdots \quad N - 1 \quad N \]

\[ s_i = +1 \quad -1 \quad -1 \quad +1 \quad +1 \quad -1 \quad +1 \]

Total Energy

\[ U = -\epsilon \left( s_1 s_2 + s_2 s_3 + s_3 s_4 + \cdots + s_{N-1} s_N \right) \]

Number of States

\[ \text{No. of states} = 2^N \]
Ising’s Solution (1D)

\[ Z = \sum_{s_1} \sum_{s_2} \ldots \sum_{s_N} e^{\beta \epsilon (s_1 s_2 + s_2 s_3 + \ldots + s_{N-1} s_N)} \]

Note

\[ \sum_{s_N} e^{\beta s_{N-1} s_N} = e^{\beta \epsilon} + e^{-\beta \epsilon} = 2 \cosh \beta \epsilon \]

independent of \( s_{N-1} \). For \( N \gg 1 \):

\[ Z = 2^N [\cosh(\beta \epsilon)]^{N-1} \approx 2^N [\cosh(\beta \epsilon)]^N \]
Ising’s Solution (1D) cont.

\[ Z = 2^N \left[ \cosh(\beta \epsilon) \right]^{N-1} \approx 2^N \left[ \cosh(\beta \epsilon) \right]^N \]

Average Energy

\[ \overline{U} = -\frac{\partial}{\partial \beta} \ln Z = -N \epsilon \tanh(\beta \epsilon) \]
2D Ising Model with $3^2 = 9$ spins

\[ U = -\varepsilon \left[ s_1 s_2 + s_2 s_3 + s_1 s_4 + s_2 s_5 + s_3 s_6 + s_4 s_5 + s_6 s_6 + s_4 s_7 + s_3 s_8 + s_6 s_9 + s_7 s_8 + s_8 s_9 \right] \]

\[ Z = \sum_{\{s\}} e^{-\beta U} \]

12 terms, $2^9 = 512$ possible states
Consider a $10 \times 10$ lattice of spins (100 elementary dipoles)

- If your computer can handle $10^9$ states per second, how long will it take to calculate $Z$?

\[
\frac{2^{100}}{10^9/\text{s}} = \frac{10^{100\log_{10}(2)}}{10^9/\text{s}} = \frac{10^{30.1}}{10^9/\text{s}} = 10^{21.1} \text{ s}
\]

In one year $t \approx \pi \times 10^7$ seconds.
A Monte Carlo approach to $\pi$

Generate $N$ pairs of random numbers and plot $(x, y)$:

Random Numbers $0 \leq x, y \leq 1$

$N = \# \text{ pairs } (x, y)$

$n = \# \text{ inside circle}$

Fraction inside $= \text{Ratio of Areas}$

$$\lim_{N \to \infty} \frac{n}{N} = \frac{\pi(1/2)^2}{1} = \frac{\pi}{4}$$
Behavior for 100 iterations

\[ \pi \approx 4 \left( \frac{79}{100} \right) = 3.16 \]
Metropolis Algorithm